Exercise 46

Show that the sum of the x- and y-intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is equal to c.

Solution

Start by differentiating both sides of the given equation with respect to x.

$$\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}\left(\sqrt{c}\right)$$

Use the chain rule to differentiate $y=y(x)$.

$$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\frac{dy}{dx}=0$$

Solve for dy/dx.

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx}=-\frac{1}{2}x^{-1/2}$$

$$\frac{dy}{dx}=-\frac{y^{1/2}}{x^{1/2}}$$

$$\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

The slope of the tangent line at the point (x_0,y_0) is then

$$m=-\sqrt{\frac{y_0}{x_0}}.$$

Use the point-slope formula to obtain the equation of the tangent line.

$$y-y_0=m(x-x_0)$$

$$y-y_0=-\sqrt{\frac{y_0}{x_0}}(x-x_0) \tag{1}$$

To determine the y-intercept, set $x=0$ in equation (1) and solve for y.

$$y-y_0=-\sqrt{\frac{y_0}{x_0}}(-x_0)$$

$$y-y_0=\sqrt{x_0}\sqrt{y_0}$$

$$y=\sqrt{x_0}\sqrt{y_0}+y_0$$

To determine the x-intercept, set $y=0$ in equation (1) and solve for x.

$$-y_0=-\sqrt{\frac{y_0}{x_0}}(x-x_0)$$

$$y_0\sqrt{\frac{x_0}{y_0}}=x-x_0$$

$$\sqrt{x_0}\sqrt{y_0}+x_0=x$$

Now add the intercepts together.

$$(\sqrt{x_0}\sqrt{y_0} + x_0) + (\sqrt{x_0}\sqrt{y_0} + y_0)$$

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

$$(\sqrt{x_0} + \sqrt{y_0})^2$$

$$\left(\sqrt{c}\right)^2$$

$$c$$

The square roots add to $\sqrt{c}$ because (x_0, y_0) lies on the curve.